

Relativistic static thin dust disks with an inner edge: An infinite family of new exact solutions

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An infinite family of new exact solutions of the Einstein vacuum equations for static and axially symmetric spacetimes is presented. All the metric functions of the solutions are explicitly computed and the obtained expressions are simply written in terms of oblate spheroidal coordinates. Furthermore, the solutions are asymptotically flat and regular everywhere, as it is shown by computing all the curvature scalars. These solutions describe an infinite family of thin dust disks with a central inner edge, whose energy densities are everywhere positive and well behaved, in such a way that their energy-momentum tensor are in fully agreement with all the energy conditions. Now, although the disks are of infinite extension, all of them have finite mass. The superposition of the first member of this family with a Schwarzschild black hole was presented previously [G. A. González and A. C. Gutiérrez-Piñeres, arXiv: 0811.3002v1 (2008)], whereas that in a subsequent paper a detailed analysis of the corresponding superposition for the full family will be presented.

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I. INTRODUCTION

The observational data supporting the existence of black holes at the nucleus of some galaxies is today so abundant, with the strongest dynamical evidence coming from the center of the Milky Way, that there is no doubt about the relevance of the study of binary systems composed by a thin disk surrounding a central black hole (see [1, 2] for recent reviews on the observational evidence). Accordingly, a lot of work has been developed in the last years in order to obtain a better understanding of the different aspects involved in the dynamics of these systems. Now, due to the presence of a black hole, the gravitational fields involved are so strong that the proper theoretical framework to analytically study this configurations is provided by the general theory of relativity. Therefore, a strong effort has been dedicated to the obtention of exact solutions of Einstein equations corresponding to thin disklike sources with a central black hole (see [3, 4] for thoroughly surveys on the subject).

Stationary and axially symmetric solutions of the Einstein equations are of obvious astrophysical importance, as they describe the exterior of equilibrium configurations of bodies in rotation. At the same time, such spacetimes are the best choice to attempt to describe the gravitational fields of disks around black holes in an exact analytical manner. So, through the years, several examples of solutions corresponding to black holes or to thin disklike sources has been obtained by many different techniques. However, due to the nonlinear character of the Einstein equations, solutions corresponding to the superposition of black holes and thin disks are not so easy to

obtain and so, until now, exact stationary solutions have not been obtained.

On the other hand, if we only consider static configurations, the line element is characterized only by two metric functions. So, in the vacuum case, the Einstein equations implies that one of the metric functions satisfies the Laplace equation whereas that the other one can be obtained by quadratures. Furthermore, as the sources are infinitesimally thin disks, the matter only enters in the form of boundary conditions for the vacuum equations. Therefore, as a consequence of the linearity of the Laplace equation, solutions corresponding to the superposition of thin disks and black holes can be, in principle, easily obtained.

However, if we consider thin disks that extend up to the event horizon, the matter located near the black hole will moves with superluminal velocities, as was shown by Lemos and Letelier [5, 6, 7]. So, in order to prevent the appearance of tachyonic matter, the thin disks must have an inner edge with a radius larger than the photonic radius of the black hole. Then, the boundary value problem for the Laplace equation is mathematically more complicated and thus only very few exact solutions had been obtained. These kind of solutions were first studied by Lemos and Letelier [6] by making a Kelvin transformation in order to invert the Morgan and Morgan [8] family of finite thin disks. Now, although the second metric function of this solution can not be analytically obtained, their main properties were extensively analyzed in a series of papers by Semerák, Žáček and Zellerin [9, 10, 11, 12, 13, 14, 15], by using numerical computation when was needed.

Besides the Lemos and Letelier inverted disks, only two other solutions for static thin disks with an inner edge have been obtained, a first one with inverted isochrone disks [16] and a second one for disks with a power-law density [17]. Also, a stationary superposition was obtained by Zellerin and Semerák [18] by

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using the Belinskii-Zakharov inverse-scattering method, but the analysis of its properties is complicated by the fact that the metric functions can not be analytically computed. Furthermore, this solution involves an unphysical supporting surface between the black-hole horizon and the disk [19]. Finally, a general class of stationary solutions was presented by Klein [20], by using the Riemann-surface techniques, with which physically acceptable black hole disk systems can in principle be found.

Now, a common feature of all the above mentioned solutions is that their metric functions can not be fully analytically computed. Thus then, the analysis of their physical and mathematical properties is very complicated. Moreover, almost all of these solutions do present singularities at the inner edge of the disk, perhaps the only exemption being the class of solutions presented by Klein [20]. Now then, the ubiquitous presence of this singularity at almost all the obtained solutions has been considered by some authors as tightly connected with the unphysical infinite thinness of the source. However, as we will show at this paper, if the corresponding boundary value problem is properly solved, it is possible to obtain a whole family of singularity free solutions.

In this paper we present an infinite family of new exact solutions for static thin dust disks with a central inner edge. These solutions describe disks whose energy densities are everywhere positive and well behaved, in such a way that their energy-momentum tensor are in fully agreement with all the energy conditions. Now, although the disks are of infinite extension, all of them have finite mass. Furthermore, the solutions are asymptotically flat and regular everywhere, as it is shown by computing all the curvature scalars. The superposition of the first member of this family with a Schwarzschild black hole was presented previously [21], whereas that in a subsequent paper a detailed analysis of the corresponding superposition for the full family will be presented.

The paper is organized as follows. First, in Sec. II, we present the formulation of the Einstein equations for static axially symmetric spacetimes with an infinitesimally thin disk as source. We also present the proper boundary conditions and their relationship with the physical quantities characterizing the sources. Then, in Sec. III, we introduce oblate spheroidal coordinates with the ranges chosen in such a way that they be naturally adapted to the geometry of a thin disk with a central inner edge. The Einstein equations are then solved and the metric functions of whole the family of solutions are explicitly computed and the obtained expressions are simply written in terms of the oblate spheroidal coordinates. The behavior of the solutions is then analysed in Sec. IV. The analysis is made by studying the behavior of all the curvature invariants for the full family as well as the behavior of the corresponding energy densities and azimuthal pressures. Also we analyze the mass densities of the disks and the finiteness of their corresponding total mass. Finally, in Sec. V, we summarize the results.

II. THE EINSTEIN EQUATIONS WITH THIN DISKLIKE SOURCES

In order to formulate the Einstein equations for static axially symmetric spacetimes with an infinitesimally thin disk as source, first we introduce coordinates $x^a = (t, \varphi, r, z)$ in which the metric tensor only depends on r and z . We assume that these coordinates are quasi-cylindrical in the sense that the coordinate r vanishes on the axis of symmetry and, for fixed z , increases monotonically to infinity, while the coordinate z , for fixed r , increases monotonically at the interval $(-\infty, \infty)$. The azimuthal angle φ ranges at the interval $[0, 2\pi)$, as usual [22]. Also we assume that there exists at the spacetime an infinitesimally thin disk, located at the hypersurface $z = 0$, in such a way that the components of the metric tensor are symmetrical functions of z and that their first z -derivatives have a finite discontinuity at $z = 0$.

Accordingly with the above considerations,

$$g_{ab}(r, -z) = g_{ab}(r, z), \quad (1)$$

in such a way that, for $z \neq 0$,

$$g_{ab,z}(r, -z) = -g_{ab,z}(r, z). \quad (2)$$

Thus then, the metric tensor is continuous at $z = 0$,

$$[g_{ab}] = g_{ab}|_{z=0^+} - g_{ab}|_{z=0^-} = 0, \quad (3)$$

whereas that the discontinuities in the derivatives of the metric tensor can be written as

$$\gamma_{ab} = [g_{ab,z}] = 2g_{ab,z}|_{z=0^+}, \quad (4)$$

where the reflection symmetry with respect to $z = 0$ has been used. So, by using the distributional approach [23, 24, 25], we can write the metric tensor as

$$g_{ab} = g_{ab}^+ \theta(z) + g_{ab}^- \{1 - \theta(z)\}, \quad (5)$$

in such a way that the Ricci tensor can be written as

$$R_{ab} = R_{ab}^+ \theta(z) + R_{ab}^- \{1 - \theta(z)\} + H_{ab} \delta(z), \quad (6)$$

where $\theta(z)$ and $\delta(z)$ are, respectively, the Heaviside and Dirac distributions with support on $z = 0$. Here g_{ab}^\pm and R_{ab}^\pm are the metric tensors and the Ricci tensors of the $z \geq 0$ and $z \leq 0$ regions, respectively, whereas that

$$H_{ab} = \frac{1}{2} \{ \gamma_a^z \delta_b^z + \gamma_b^z \delta_a^z - \gamma_c^c \delta_a^z \delta_b^z - g^{zz} \gamma_{ab} \}, \quad (7)$$

where all the quantities are evaluated at $z = 0^+$.

Then, in agreement with (6), the energy-momentum tensor must be expressed as

$$T_{ab} = T_{ab}^+ \theta(z) + T_{ab}^- \{1 - \theta(z)\} + Q_{ab} \delta(z), \quad (8)$$

where T_{ab}^\pm are the energy-momentum tensors for the $z \geq 0$ and $z \leq 0$ regions, respectively, and Q_{ab} gives the part

of the energy-momentum tensor corresponding to the disk source. Accordingly, the Einstein equations, in geometrized units such that $c = 8\pi G = 1$, are equivalent to the system

$$R_{ab}^{\pm} - \frac{1}{2}g_{ab}R^{\pm} = T_{ab}^{\pm}, \quad (9)$$

$$H_{ab} - \frac{1}{2}g_{ab}H = Q_{ab}, \quad (10)$$

where $H = g^{ab}H_{ab}$ and, again, all the quantities are evaluated at $z = 0^+$. Now then, when the thin disk is the only source of the gravitational field, so that $T_{ab}^{\pm} = 0$, equation (9) reduces to the Einstein vacuum equations

$$R_{ab}^{\pm} = 0, \quad (11)$$

for the $z \geq 0$ and $z \leq 0$ regions, respectively. Thus, in order to obtain solutions with a thin disk as source, we must solve the system (11) by using in equation (10), as boundary conditions, the values of Q_{ab} that describe properly the matter content of the disk.

Now, in order to obtain explicit forms for the vacuum Einstein equations and the boundary conditions, we will take the metric tensor as given by the Weyl line element, written as [26]

$$ds^2 = -e^{2\Phi}dt^2 + e^{-2\Phi}[r^2d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)], \quad (12)$$

where Φ and Λ are continuous functions of r and z . Furthermore, we will assume that Φ and Λ are even functions of z ,

$$\Phi(r, -z) = \Phi(r, z), \quad (13a)$$

$$\Lambda(r, -z) = -\Lambda(r, z), \quad (13b)$$

in such a way that their first z -derivatives are odd functions of z ,

$$\Phi_{,z}(r, -z) = -\Phi_{,z}(r, z), \quad (14a)$$

$$\Lambda_{,z}(r, -z) = -\Lambda_{,z}(r, z), \quad (14b)$$

which we shall require that do not vanish at $z = 0$. So, the vacuum Einstein equations (11) are equivalent to the system

$$(r\Phi_{,r})_{,r} + (r\Phi_{,z})_{,z} = 0, \quad (15a)$$

$$\Lambda_{,r} = r(\Phi_{,r}^2 - \Phi_{,z}^2), \quad (15b)$$

$$\Lambda_{,z} = 2r\Phi_{,r}\Phi_{,z}, \quad (15c)$$

where (15a) is the usual Laplace equation for an axially symmetric source in cylindrical coordinates, whereas that the integrability condition for the overdetermined system (15b) - (15c) is granted when Φ is a solution of (15a), in

such a way that Λ can be obtained by quadratures given a solution for Φ .

On the other hand, equation (10) implies that the boundary conditions reduce to

$$2e^{2(\Phi-\Lambda)}[\Lambda_{,z} - 2\Phi_{,z}] = Q_t^t, \quad (16a)$$

$$2e^{2(\Phi-\Lambda)}\Lambda_{,z} = Q_{\varphi}^{\varphi}, \quad (16b)$$

where all the quantities are evaluated at $z = 0^+$, and that Q_{ab} must have only two nonzero components. So, by using the orthonormal tetrad

$$V^a = e^{-\Phi} \delta_t^a, \quad (17a)$$

$$W^a = e^{\Phi} \delta_{\varphi}^a / r, \quad (17b)$$

$$X^a = e^{\Phi-\Lambda} \delta_r^a, \quad (17c)$$

$$Y^a = e^{\Phi-\Lambda} \delta_z^a, \quad (17d)$$

we can write the surface energy-momentum tensor Q_{ab} in the canonical form

$$Q_{ab} = \epsilon V_a V_b + p W_a W_b, \quad (18)$$

where ϵ and p are, respectively, the energy density and the azimuthal pressure of the disk. In terms of these quantities, the boundary conditions can be written as

$$2e^{2(\Phi-\Lambda)}[2\Phi_{,z} - \Lambda_{,z}] = \epsilon, \quad (19)$$

$$2e^{2(\Phi-\Lambda)}\Lambda_{,z} = p. \quad (20)$$

Finally, by using (15c), we can cast these conditions as

$$4e^{2(\Phi-\Lambda)}[1 - r\Phi_{,r}]\Phi_{,z} = \epsilon, \quad (21)$$

$$4e^{2(\Phi-\Lambda)}r\Phi_{,r}\Phi_{,z} = p, \quad (22)$$

where, as before, all the quantities are evaluated at $z = 0^+$.

As we can see from the above expressions, the more general energy-momentum tensor that is compatible with the line element (12) and the boundary conditions (10), corresponds to a thin disklike source that only have a nonzero energy density and a nonzero azimuthal pressure. In agreement with this, instead of give specific prescriptions for the energy density and the azimuthal pressure, the Einstein equations will be solved only by requiring that these two quantities will be different from zero at the surface of a disk with an inner edge. Then, after a given solution be obtained, it can be used in order to obtain, from the boundary conditions, the corresponding expressions for the energy density and the azimuthal pressure. Therefore, the solution will correspond to the

more general static thin disk with an inner edge that can be obtained by exactly solving the Einstein equations.

Accordingly, in order to obtain a solution representing a thin disk located in the hypersurface $z = 0$, with a circular central inner edge of radius a , we only need to impose that

$$\Phi_{,z}(r, 0^+) = \begin{cases} 0 & ; 0 \leq r \leq a, \\ f(r) & ; r \geq a, \end{cases} \quad (23)$$

with $f(r)$ any arbitrary function. Then, only after we find the more general solution, we will impose additional conditions in order to have a physically reasonable behavior. So, in order to have an asymptotically flat spacetime, we will require that

$$\lim_{R \rightarrow \infty} \Phi(r, z) = 0, \quad (24a)$$

$$\lim_{R \rightarrow \infty} \Lambda(r, z) = 0, \quad (24b)$$

where $R^2 = r^2 + z^2$. Also, in order to have regularity at the symmetry axis, we will require that

$$\Phi(0, z) < \infty, \quad (25a)$$

$$\Lambda(0, z) < \infty. \quad (25b)$$

We also will require that

$$f(r) \geq 0, \quad (26a)$$

$$0 \leq r\Phi_{,r} \leq 1, \quad (26b)$$

in order that the energy density and the azimuthal pressure be positive everywhere.

Finally, we need to check the finiteness of the total mass of the disks. So, in order to do this, first we take the mass density μ of the disks as defined by

$$\frac{\mu}{2} = (Q_{ab} - \frac{1}{2}g_{ab}Q)V^aV^b, \quad (27)$$

where $Q = g^{ab}Q_{ab}$ and, as before, all the quantities are evaluated at $z = 0^+$. Accordingly, by using (16) and (18), we have that the mass density reduces to

$$\mu = \epsilon + p. \quad (28)$$

On the other hand, the total mass of the disks is given by

$$M = \int \mu d\Sigma = \int_0^{2\pi} \int_a^\infty \mu e^{\Lambda-2\Phi} r dr d\varphi, \quad (29)$$

where $d\Sigma = e^{\Lambda-2\Phi} r dr d\varphi$ is the area element on the disk surface. So, by using (28), we can write the total mass as

$$M = 2\pi \int_a^\infty (\epsilon + p) e^{\Lambda-2\Phi} r dr, \quad (30)$$

where the integration on φ has been made.

III. SOLUTION OF THE EINSTEIN EQUATIONS

In order to solve the Einstein vacuum equations, first we must solve the boundary value problem for Φ . However, due to the nature of the boundary conditions (14), it is convenient to look for a different coordinate system that be naturally adapted to the geometry of the desired source. Accordingly, we introduce the oblate spheroidal coordinates as defined through the relations

$$r^2 = a^2(1+x^2)(1-y^2), \quad (31a)$$

$$z = axy, \quad (31b)$$

where $-\infty < x < \infty$ and $0 \leq y \leq 1$. So, when $x = 0$ we have that $z = 0$ and $0 \leq r \leq a$, whereas that when $y = 0$ we have that $z = 0$ and $r \geq a$. Furthermore, as x changes sign on crossing the surface $y = 0$, but do not changes in absolute value, this coordinate has a finite discontinuity when $y = 0$. Thus then, an even function of x is a continuous function everywhere but has a discontinuous normal derivative at $y = 0$. On the other hand, y is continuous everywhere. Therefore, the surface $y = 0$ describes a thin disk with an inner edge of radius a , whereas that the surface $x = 0$ describes the vacuum hole inside this edge.

In the oblate spheroidal coordinates, the Weyl line element (12) can be rewritten as

$$ds^2 = -e^{2\Phi} dt^2 + a^2(1+x^2)(1-y^2)e^{-2\Phi} d\varphi^2 + a^2(x^2+y^2)e^{2(\Lambda-\Phi)} \left[\frac{dx^2}{1+x^2} + \frac{dy^2}{1-y^2} \right], \quad (32)$$

in such a way that the Einstein vacuum equations reduce to

$$[(1+x^2)\Phi_{,x}]_{,x} + [(1-y^2)\Phi_{,y}]_{,y} = 0, \quad (33)$$

the Laplace equation in oblate spheroidal coordinates, and the overdetermined system

$$\Lambda_{,x} = (1-y^2) [x(1+x^2)\Phi_{,x}^2 - x(1-y^2)\Phi_{,y}^2 - 2y(1+x^2)\Phi_{,x}\Phi_{,y}] / (x^2+y^2), \quad (34)$$

$$\Lambda_{,y} = (1+x^2) [y(1+x^2)\Phi_{,x}^2 - y(1-y^2)\Phi_{,y}^2 + 2x(1-y^2)\Phi_{,x}\Phi_{,y}] / (x^2+y^2), \quad (35)$$

whose integrability again is granted by equation (33).

On the other hand, by using (31a) and (31b), is easy to see that

$$\Phi_{,z}(r, 0) = \begin{cases} \Phi_{,x}(0, y)/ay & ; 0 \leq r \leq a, \\ \Phi_{,y}(x, 0)/ax & ; r \geq a. \end{cases} \quad (36)$$

Accordingly, as the reflection symmetry of the solutions

implies that

$$\Phi(-x, y) = \Phi(x, y), \quad (37a)$$

$$\Phi_{,x}(-x, y) = -\Phi_{,x}(x, y), \quad (37b)$$

the conditions (23) are equivalent to

$$\Phi_{,x}(0, y) = 0, \quad (38a)$$

$$\Phi_{,y}(x, 0) = F(x); \quad x \geq 0, \quad (38b)$$

with $F(x)$ any arbitrary function. The general solution of equation (33) with these boundary conditions is given by [27]

$$\Phi(x, y) = \sum_{n=0}^{\infty} [A_{2n}P_{2n}(y) + B_{2n}Q_{2n}(y)]p_{2n}(x), \quad (39)$$

where A_{2n} and B_{2n} are constants, $P_{2n}(y)$ and $Q_{2n}(y)$ are the Legendre polynomials and the Legendre functions of the second kind, respectively, and $p_{2n}(x) = i^{-2n}P_{2n}(ix)$. Therefore, all the solutions of the Einstein vacuum equations for static spacetimes with any axially symmetric source as the considered here, are obtained by taking for the metric function $\Phi(x, y)$ any particular choice of the above general solution, or expressions obtained from these solutions by means of linear operations.

Now, in terms of the oblate spheroidal coordinates, condition (24a) is written as

$$\lim_{x \rightarrow \infty} \Phi(x, y) = 0, \quad (40)$$

whereas condition (25a) is written as

$$\Phi(x, 1) < \infty. \quad (41)$$

So, due to the behavior of the Legendre functions, it is clear that it is not possible to fulfill the physical conditions (24) and (25) with any particular choice of the general solution (39). However, by considering only the first term of the series,

$$\Phi_0(x, y) = A_0 + B_0Q_0(y), \quad (42)$$

we obtain a solution that is regular for all $y \neq 1$. Then, if we take $A_0 = 0$, this solution can be written as

$$\Phi_0(x, y) = \frac{\alpha}{2} \ln \left[\frac{1+y}{1-y} \right], \quad (43)$$

where α is an arbitrary constant, in such a way that a direct integration of (34)-(35) gives

$$\Lambda_0(x, y) = \frac{\alpha^2}{2} \ln \left[\frac{1-y^2}{x^2+y^2} \right], \quad (44)$$

where the integration constant has been taken as zero. As we can see, this solution is not asymptotically flat

neither regular at the symmetry axis. Nevertheless, by using (19) and (20), we obtain for the energy density and the azimuthal pressure the expressions

$$\epsilon = (4\alpha/a)x^{2\alpha^2-1}, \quad (45)$$

$$p = 0, \quad (46)$$

in such a way that, if $\alpha > 0$, the disk satisfy all the energy contions [28]. However, for any value of $\alpha^2 \neq \frac{1}{2}$, the energy density increases without limit, may be at infinity or at the inner edge of the disk, whereas that for $\alpha^2 = \frac{1}{2}$ the energy density is everywhere constant, so that, in any case, the total mass of the disk will be infinite.

On the other hand, although the previous solution has not a physically acceptable behavior, we can use it as the starting point to generate new and well behaved solutions. In order to do this, we consider the oblate spheroidal coordinates not only as functions of the cylindrical coordinates (r, z) , but also as parametrically dependents of the radius a ,

$$x = x(r, z; a), \quad (47a)$$

$$y = y(r, z; a). \quad (47b)$$

Accordingly, by considering also the metric function Φ as dependent of a ,

$$\Phi = \Phi(r, z; a), \quad (48)$$

we can obtain a family of new solutions by applying the linear operation

$$\Phi_{n+1}(r, z; a) = \frac{\partial \Phi_n(r, z; a)}{\partial a}, \quad (49)$$

where n is an integer, $n \geq 0$.

Thus then, by starting with the “seed solution” $\Phi_0(x, y)$, by means of the previous procedure it is generated a family of new solutions that can be written in the simple form

$$\Phi_n(r, z; a) = \Phi_n(x, y) = \frac{\alpha y F_n(x, y)}{a^n (x^2 + y^2)^{2n-1}}, \quad (50)$$

for $n \geq 1$, where the $F_n(x, y)$ are polynomial functions, with highest degree $4n-4$, of which only we present below the first three,

$$F_1 = 1,$$

$$F_2 = x^4 + 3x^2(1-y^2) - y^2,$$

$$F_3 = 3x^6(3-5y^2) + 5x^4(6y^4-11y^2+3) - x^2y^2(3y^4-31y^2+30) - y^4(y^2-3),$$

but all of them can be easily obtained by means of (49). So, is easy to see that

$$\lim_{x \rightarrow \infty} \Phi_n(x, y) = 0, \quad (51)$$

and that

$$\Phi_n(x, 1) < \infty, \quad (52)$$

in fully agreement with conditions (24a) and (25a).

Now, in order to obtain the correspondig metric functions $\Lambda_n(r, z; a)$, we make the integration

$$\Lambda_n(r, z; a) = \Lambda_n(x, y) = \int_1^y \Lambda_{,y}(x, y) dy, \quad (53)$$

by taking $\Lambda_n(x, 1) = 0$ in order to grant regularity at the axis. So, by using (50) in (35), the obtained solutions can be written in the simple form

$$\Lambda_n(x, y) = \frac{\alpha^2 (2n-2)! (y^2-1) A_n(x, y)}{4^n a^{2n} (x^2 + y^2)^{4n}}, \quad (54)$$

for $n \geq 1$, where the $A_n(x, y)$ are polynomial functions, off highest degree $8n-2$, of which we present here only the first three,

$$A_1 = x^4(9y^2 - 1) + 2x^2y^2(y^2 + 3) + y^4(y^2 - 1),$$

$$\begin{aligned} A_2 = & 2x^{12}(9y^2 - 1) - 4x^{10}(51y^4 - 41y^2 + 2) \\ & + x^8(735y^6 - 1241y^4 + 419y^2 - 9) - x^6y^2(132y^6 \\ & - 1644y^4 + 1604y^2 - 252) + x^4y^4(84y^6 - 384y^4 \\ & + 1266y^2 - 630) + 4x^2y^6(6y^6 + 6y^4 - 39y^2 + 63) \\ & + 3y^8(y^6 + y^4 + y^2 - 3), \end{aligned}$$

$$\begin{aligned} A_3 = & 3x^{16}(1225y^6 - 1275y^4 + 315y^2 - 9) \\ & - 24x^{14}(980y^8 - 2095y^6 + 1205y^4 - 189y^2 + 3) \\ & + 2x^{12}(24255y^{10} - 89475y^8 + 98472y^6 - 36316y^4 \\ & + 3473y^2 - 25) - 12x^{10}y^2(1835y^{10} - 16665y^8 \\ & + 34716y^6 - 25292y^4 + 6001y^2 - 275) \\ & + 6x^8y^4(900y^{10} - 11946y^8 + 50563y^6 - 69397y^4 \\ & + 33365y^2 - 4125) + 8x^6y^6(125y^{10} + 926y^8 \\ & - 9079y^6 + 24639y^4 - 22290y^2 + 5775) \\ & + 6x^4y^8(55y^{10} + 29y^8 + 764y^6 - 4808y^4 + 8469y^2 \\ & - 4125) + 12x^2y^{10}(5y^{10} + 5y^8 + 80y^4 - 301y^2 \\ & + 275) + y^{12}(5y^{10} + 5y^8 + 5y^6 + y^4 + 34y^2 - 50), \end{aligned}$$

but all of them can be obtained as a result of compute the integral (53). Accordingly, we have that

$$\lim_{x \rightarrow \infty} \Lambda_n(x, y) = 0, \quad (55)$$

and that

$$\Lambda_n(x, 1) = 0, \quad (56)$$

in fully agreement with conditions (24b) and (25b).

IV. BEHAVIOR OF THE SOLUTIONS

As we can see from the expressions at previous section, by using (49) was generated an infinite family of asymptotically flat solutions, all of them with a regular behavior at the symmetry axis. On the other hand, it seems to be that there is a singularity at the inner edge of the disks, when $x = 0$ and $y = 0$. However, is easy to see that this apparent singularity is only a coordinate singularity. Indeed, as it is well known [26, 29], the invariant characterization of the curvature must be in terms of 14 scalars constructed from the Riemann tensor, R_{abcd} , and the metric tensor, g_{ab} . So, in order to determine the nature of the aparent singularities, we must to compute all the curvature invariants for the family of solutions.

Now, for any solution of the Einstein vacuum equations, the curvature invariants are the 10 vanishing components of the Ricci tensor, $R_{ab} = 0$, plus the four scalars

$$\mathcal{K}_I = R^{abcd} R_{abcd},$$

$$\mathcal{K}_{II} = R^{ab}{}_{kl} R^{klcd} R_{abcd},$$

$$\mathcal{K}_{III} = \frac{\epsilon^{ab}{}_{kl} R^{klcd} R_{abcd}}{\sqrt{-g}},$$

$$\mathcal{K}_{IV} = \frac{\epsilon^{ab}{}_{kl} R^{kl}{}_{mn} R^{mncd} R_{abcd}}{\sqrt{-g}},$$

where $g = \det g_{ab}$ and ϵ^{abcd} is the Levi-Civita symbol. Furthermore, for any Weyl solution the last two invariants vanishes identically, so that we only need to compute \mathcal{K}_I and \mathcal{K}_{II} .

By using the expressions (50) and (54) for $\Phi_n(x, y)$ and $\Lambda_n(x, y)$, we can cast the curvature scalars as

$$\mathcal{K}_{In} = -\frac{16\alpha^2 e^{4(\Phi_n - \Lambda_n)} N_{In}(x, y)}{a^{6n+4} (x^2 + y^2)^{12n}}, \quad (57a)$$

$$\mathcal{K}_{II n} = \frac{48\alpha^3 e^{6(\Phi_n - \Lambda_n)} N_{II n}(x, y)}{a^{8n+6} (x^2 + y^2)^{16n}}, \quad (57b)$$

where $N_{In}(x, y)$ and $N_{II n}(x, y)$ are polynomial functions, of highest degree $24n-6$ and $32n-9$, respectively, that we do not write here explicitly due to their long size, but that all of them vanishes at the inner edge of the disks,

$$N_{In}(0, 0) = N_{II n}(0, 0) = 0. \quad (58)$$

Moreover, is easy to see that, in any neighborhood around $(0, 0)$, the difference between $\Phi_n(x, y)$ and $\Lambda_n(x, y)$ behaves as

$$\Phi_n - \Lambda_n \approx -\frac{\alpha^2}{a^{2n} (x^2 + y^2)^{2n}}, \quad (59)$$

in such a way that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{4(\Phi_n - \Lambda_n)}}{(x^2 + y^2)^{12n}} = 0, \quad (60a)$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{6(\Phi_n - \Lambda_n)}}{(x^2 + y^2)^{16n}} = 0, \quad (60b)$$

and the limits exist, whatever be the path chosen to approach the point $(0, 0)$. Accordingly, we have that

$$\lim_{(x,y) \rightarrow (0,0)} \mathcal{K}_{In}(x, y) = 0, \quad (61a)$$

$$\lim_{(x,y) \rightarrow (0,0)} \mathcal{K}_{IIn}(x, y) = 0, \quad (61b)$$

and thus the curvature is regular at the inner edge of the disks.

Now, in order to analyze the physical behavior of the sources, we will compute the energy density and the azimuthal pressure for this family of disks. So, by using (31a) and (31b), we can see that

$$\Phi_{,r}(r, 0) = \left[\frac{\sqrt{1+x^2}}{ax} \right] \Phi_{,x}(x, 0), \quad r \geq a, \quad (62)$$

and, by using (50), is easy to prove that

$$\Phi_{n,x}(x, 0) = 0, \quad n \geq 1. \quad (63)$$

Then, from (22), we can see that

$$p_n = 0. \quad (64)$$

That is, all the disks of the family have zero azimuthal pressure.

On the other hand, by using equations (21), (36) and (50), the surface energy density of the disks can be written as

$$\epsilon_n(x) = \frac{4\alpha E_n(x)}{a^{n+1}x^{2n+1}} \exp \left\{ -\frac{\alpha^2(2n-2)!B_n(x)}{2^{2n-1}a^{2n}x^{4n}} \right\}, \quad (65)$$

where $x \geq 0$ and the $E_n(x)$ are positive definite polynomials of degree $2k$, with $k = (n-1)/2$ for odd n and $k = n/2$ for even n , of which we only will write below the first three,

$$E_1(x) = 1,$$

$$E_2(x) = x^2 + 3,$$

$$E_3(x) = 3(x^2 + 5),$$

whereas that the $B_n(x)$ are positive definite polynomials of degree $4k$, with $k = (n-1)/2$ for odd n and $k = n/2$ for even n , the first three of them given by

$$B_1(x) = 1,$$

$$B_2(x) = 2x^4 + 8x^2 + 9,$$

$$B_3(x) = 27x^4 + 72x^2 + 50.$$

From the above expressions we can see that, by taking $\alpha > 0$, the energy density of the disks will be everywhere positive,

$$\epsilon_n(x) \geq 0. \quad (66)$$

So that, as the azimuthal pressure is zero, we have an infinite family of dust disks that are in fully agreement with all the energy conditions. Also is easy to see that, for any value of n , we have that

$$\epsilon_n(0) = 0, \quad (67a)$$

$$\lim_{x \rightarrow \infty} \epsilon_n(x) = 0. \quad (67b)$$

That is, the energy density of the disks is zero at their inner edge and vanishes at infinite. Furthermore, as the azimuthal pressure is zero, the mass density of the disks reduces to their energy density,

$$\mu_n(x) = \epsilon_n(x), \quad (68)$$

so that its behavior is the same as the energy density.

Now, in order to show the behavior of the energy densities, we plot the dimensionless surface energy densities $\tilde{\epsilon}_n = a\epsilon_n$ as functions of the dimensionless radial coordinate $\tilde{r} = r/a$. So, in Figure 1, we plot $\tilde{\epsilon}_n$ as a function of \tilde{r} for the first three disks of the family, with $n = 1, 2$ and 3 , for different values of the parameter $\tilde{\alpha}_n = \alpha/a^n$. Then, for each value of n , we take $\tilde{\alpha}_n = 0.5, 1, 1.5, 2, 2.5, 3, 3.5$ and 4 . The first curve on left corresponds to $\tilde{\alpha}_n = 0.5$, whereas that the last curve on right corresponds to $\tilde{\alpha}_n = 4$. As we can see, in all the cases the surface energy density is everywhere positive, having a maximum near the inner edge of the disks, and then rapidly decreasing as \tilde{r} increases. We can also see that, for a fixed value of n , as the value of $\tilde{\alpha}_n$ increases, the value of the maximum diminishes and moves towards increasing values of \tilde{r} . The same behavior is observed for a fixed value of $\tilde{\alpha}_n$ and increasing values of n .

On the other hand, by using (30), (31a) and (68), the total mass of the disks can be expressed as

$$M_n = 2\pi a \int_0^\infty f_n(x) dx, \quad (69)$$

where

$$f_n(x) = x e^{-\Lambda_n(x,0)} \epsilon_n(x). \quad (70)$$

So, from (54), (65) and (70), is easy to see that

$$\lim_{x \rightarrow \infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{x \rightarrow \infty} \frac{\epsilon_{n+1}(x)}{\epsilon_n(x)}, \quad (71a)$$

$$= \lim_{x \rightarrow \infty} \frac{E_{n+1}(x)}{ax^2 E_n(x)}, \quad (71b)$$

$$= \begin{cases} \frac{1}{a} & ; n = 2k+1, \\ 0 & ; n = 2k+2, \end{cases} \quad (71c)$$

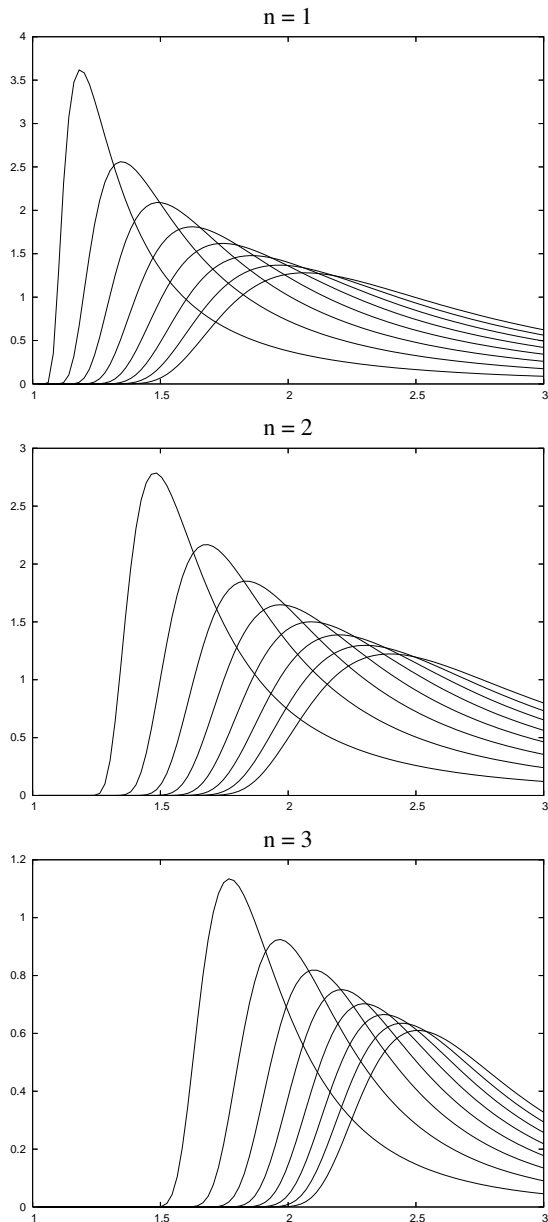


FIG. 1: Surface energy density $\tilde{\epsilon}_n$ as a function of \tilde{r} for the first three disks of the family, with $\tilde{\alpha}_n = 0.5, 1, 1.5, 2, 2.5, 3, 3.5$ and 4 . For each value of n , the first curve on left corresponds to $\tilde{\alpha} = 0.5$, whereas that the last curve on right corresponds to $\tilde{\alpha} = 4$.

with $k \geq 0$. Therefore, by the limit comparison test for improper integrals, the convergence of M_{n+1} is granted if M_n is convergent and thus we only need to test the convergence of M_1 . Indeed, a simple computation gives

$$M_1 = 2\pi\sqrt{2a\alpha} \Gamma(1/4), \quad (72)$$

by granting so the convergence of all the mass integrals (69). Accordingly, although the disks are of infinite extension, all of them have finite mass.

V. CONCLUDING REMARKS

We presented an infinite family of asymptotically flat and everywhere regular exact solutions of the Einstein vacuum equations. These solutions describe an infinite family of thin dust disks with a central inner edge, whose energy densities are everywhere positive and well behaved, in such a way that their energy-momentum tensor are in fully agreement with all the energy conditions. Moreover, although the disks are of infinite extension, all of them have finite mass. Now, as all the metric functions of the solutions are explicitly computed, these are the first fully integrated explicit exact solutions for such kind of thin disk sources. Furthermore, their relative simplicity when expressed in terms of oblate spheroidal coordinates, makes it very easy to study different dynamical aspects, like the motion of particles inside and outside the disks and the stability of the orbits.

Now, besides their importance as a new family of exact and explicit solutions of the Einstein vacuum equations, the main importance of this family of solutions is that they can be easily superposed with the Schwarzschild solution in order to describe binary systems composed by a thin disk surrounding a central black hole. Indeed, the superposition of the first member of this family with a Schwarzschild black hole already has been done, and was previously presented in [21], whereas that in a subsequent paper a detailed analysis of the corresponding superposition for the full family will be presented. Accordingly, like was the family presented here, its superposition with the Schwarzschild black hole will be the first family of explicitly integrated exact solutions for this superposition of sources.

Finally, it is worth to mention an interesting feature of this family of solutions. As in Newtonian theory the gravitational potential is given by the solution of the boundary value problem for the Laplace equation, we can consider the $\Phi_n(r, z; a)$ as a family of Newtonian gravitational potentials of thin disklike sources with an inner edge, whose Newtonian mass densities are given by

$$\sigma_n(x) = \frac{2\alpha E_n(x)}{a^{n+1}x^{2n+1}}, \quad (73)$$

clearly diverging at the edge of the disks. Accordingly, we can conclude that there are not regular solutions within the Newtonian theory that properly describe the gravitational field of a thin disk with an inner edge, whereas that this kind of source can be properly described, by means of regular and asymptotically flat solutions, within the general relativistic gravitation theory.

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